The electrodynamics of doubly anisotropic media with non-parallel principal axes of permittivity and permeability

1974 J. Phys. A: Math. Nucl. Gen. 7 600

(http://iopscience.iop.org/0301-0015/7/5/010)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 5, 1974. Printed in Great Britain. © 1974

The electrodynamics of doubly anisotropic media with non-parallel principal axes of permittivity and permeability

G P Sastry

Department of Physics, Indian Institute of Technology, Kharagpur, India

Received 3 September 1973

Abstract. Considerations based on the symmetry inherent in Maxwell's electrodynamics enable us to tackle problems involving doubly anisotropic media with non-parallel principal axes of ϵ and μ. It is shown that one can pass on from the results of single anisotropy to the more general double anisotropy with the aid of a set of substitutions. These are then applied to obtain expressions for the radiation cones and energy loss of a charge moving uniformly in a generalized uniaxial medium.

1. Introduction

Studies on the optics of anisotropic media generally ignore the tensor character of magnetic permeability and assume that $\mu = 1$. This is justified in the range of optical frequencies, where the time periods of electromagnetic oscillations are much shorter than relaxation times of magnetic phenomena (Landau and Lifshitz 1960, Born and Wolf 1965). At microwave frequencies on the other hand, doubly anisotropic materials find frequent application. For instance, certain magnetic complexes imbedded in crystals of high dielectric constant as host lattices (eg chromium-doped titania) appear to have distinct advantages as maser materials (Gerritsen *et al* 1960). In such situations, the tensors ϵ and μ often have their principal frames inclined to one another (Lewandowski 1971). A comprehensive treatment of the electrodynamics of such a general medium must of necessity be considerably involved. Yet, the remarkable symmetry with which electrical and magnetic quantities enter Maxwell's equations can be expected to mitigate the complexity of even this general case. This symmetry has been clearly brought out and exploited by Majumdar (1973) to reduce the problem of Čerenkov radiation in a doubly anisotropic medium to that in a singly anisotropic medium with $\mu = 1$.

Čerenkov radiation extends continuously into the low frequencies and is of interest in the production of microwaves (Jelley 1958). In this context, the use of ferrites (which generally have a tensor permeability) has been advocated in order to augment the generated power by a large factor (Lashinsky 1956). Čerenkov radiation in doubly anisotropic media has therefore been the object of considerable study (Muzikar and Pafamov 1961, Majumdar and Pal 1970, 1973). These studies, however, confine themselves to the special case of coincident principal frames of ϵ and μ.

In the present paper, we consider a general doubly anisotropic medium with non-parallel principal frames and show that the field and energy loss of a charge moving uniformly in it can be obtained by effecting a set of simple substitutions in the results for the case of single anisotropy. The mathematical situation is highly symmetric in $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$, and one could start either with pure magnetic anisotropy or pure electric anisotropy. The technique of this substitution is illustrated by calculating the energy loss of a uniformly moving charge in a generalized uniaxial medium (Majumdar 1973). The solution of this apparently intractable problem turns out in the end to be surprisingly simple. The Čerenkov cone in this instance is made up of two distinct second degree surfaces whose equations are derived using the same set of transformations.

Again the Čerenkov cone is connected to the Fresnel wave surface in the medium by a very simple transformation (Majumdar and Pal 1973). The wave surface being a picturesque description of the characteristics of wave propagation in a variety of situations (O'Dell 1970), one is led to expect that a similar reduction of double anisotropy to single anisotropy should be a feature common to several other aspects of the optics of general doubly anisotropic media.

1.1. Explanation of the symbols used

All quantities referred to the $\boldsymbol{\mu}$ and $\boldsymbol{\epsilon}$ frames are indicated with the superposed signs ($\bar{}$) and ($\hat{}$) respectively.

X^0, X are the coordinates of a field point in the rest frame of the medium and of the charge respectively.

$\mathbf{P}$ is the orthogonal matrix transforming $\bar{X}$ to $\hat{X}$.

$\bar{\mathbf{R}}$, $\hat{\mathbf{R}}$ are the transformations from the X^0 frame to the $\boldsymbol{\mu}$ and $\boldsymbol{\epsilon}$ frames respectively.

$\boldsymbol{\epsilon}$, $\boldsymbol{\mu}$ are the permittivity and the permeability tensors in the X^0 frame.

$\boldsymbol{\lambda} = (\boldsymbol{\mu})^{-1}$ and $\boldsymbol{\eta} = (\boldsymbol{\epsilon})^{-1}$.

$\bar{\mathbf{D}}$ is the diagonal matrix with $\bar{D}_{ij} = 0$ for $i \neq j$ and $\bar{D}_{ii} = \sqrt{\lambda_i}$.

$\hat{\mathbf{D}}$ is the diagonal matrix with $\hat{D}_{ij} = 0$ for $i \neq j$ and $\hat{D}_{ii} = \sqrt{\epsilon_i}$.

2. Symmetric reduction to single anisotropy

We choose a coordinate frame with the x_1^0 axis along the direction of uniform motion of the charge. In an inertial frame at rest with the charge, the Fourier integral for the scalar potential is given by (Majumdar 1973)

$$\Phi = \frac{e\beta^2\gamma^2}{8\pi^3} \int \frac{[a_{11}(\boldsymbol{K})]\, e^{i\boldsymbol{K}.\boldsymbol{X}}\, d^3\boldsymbol{K}}{A(\boldsymbol{K})} \tag{1}$$

where

$$A(\boldsymbol{K}) = [\lambda_{\alpha\beta}]K_\alpha K_\beta \epsilon_{\mu\nu} K_\mu K_\nu + (\boldsymbol{\epsilon\lambda\epsilon})_{\alpha\beta} K_\alpha K_\beta - (\boldsymbol{\lambda\epsilon})_{\gamma\gamma} \epsilon_{\alpha\beta} K_\alpha K_\beta + |\boldsymbol{\epsilon}| \tag{2}$$

and

$$[a_{ij}(\boldsymbol{K})] = [\lambda_{\alpha\beta}]K_\alpha K_\beta K_i K_j + (\boldsymbol{\epsilon\lambda})_{\alpha i} K_\alpha K_j + (\boldsymbol{\epsilon\lambda})_{\alpha j} K_\alpha K_i - \epsilon_{\alpha\beta} K_\alpha K_\beta \lambda_{ij} - (\boldsymbol{\epsilon\lambda})_{\alpha\alpha} K_i K_j + [\epsilon_{ij}]. \tag{3}$$

In these expressions, valid in a general biaxial medium with non-parallel principal axes of $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$, $\boldsymbol{\lambda}$ is the reciprocal permeability and K_1, ω and ϵ_{ij} stand for γK_1, $\gamma K_1 v$ and $(\omega^2/c^2)\epsilon_{ij}$ respectively. Square brackets signify the cofactors of the matrix elements they enclose. In order to simplify the expressions for $A(\boldsymbol{K})$ and $[a_{ij}(\boldsymbol{K})]$, we now subject the vectors $\boldsymbol{K}$ and $\boldsymbol{X}$ simultaneously to a series of transformations such that the scalar product $\boldsymbol{K}\,.\,\boldsymbol{X}$ remains invariant.

(i) Let the coordinates of a point in the X^0 frame be connected to those in the principal permeability ($\boldsymbol{\mu}$) frame by $X^0 = \bar{\mathbf{R}}\bar{X}$. We now make the substitutions $|\boldsymbol{\lambda}|^{1/2}\bar{\boldsymbol{K}} = \bar{\mathbf{D}}\bar{\boldsymbol{K}}'$ and $\bar{\mathbf{D}}\bar{\boldsymbol{\epsilon}}\bar{\mathbf{D}} = \bar{\boldsymbol{\epsilon}}'$ where $\bar{D}_{ij} = 0$ for $i \neq j$ and $\bar{D}_{ii} = \sqrt{\lambda_i}$. Let $\bar{\mathbf{S}}$ be the orthogonal matrix that makes $\bar{\boldsymbol{\epsilon}}'$ diagonal by $\boldsymbol{\epsilon}'' = \bar{\mathbf{S}}^{-1}\bar{\boldsymbol{\epsilon}}'\bar{\mathbf{S}}$ (Majumdar 1973). With the transformation $\bar{\boldsymbol{K}}' = \bar{\mathbf{S}}\bar{\boldsymbol{K}}''$, we have,

$$A''(\bar{\boldsymbol{K}}'') = \left(\bar{K}''^2 - \sum \epsilon''_\gamma\right)\sum \epsilon''_\alpha \bar{K}''^2_\alpha + \sum \epsilon''^2_\alpha \bar{K}''^2_\alpha + |\boldsymbol{\epsilon}''| = f(\boldsymbol{\epsilon}'', \bar{\boldsymbol{K}}'') \tag{4}$$

and

$$[a_{ij}(\bar{\boldsymbol{K}}'')]'' = \left(\bar{K}''^2 + \epsilon''_i + \epsilon''_j - \sum \epsilon''_\alpha\right)\bar{K}''_i\bar{K}''_j - \delta_{ij}\sum \epsilon''_\alpha \bar{K}''^2_\alpha + [\epsilon''_{ij}]. \tag{5}$$

It is then evident that these expressions for $A''(\bar{\boldsymbol{K}}'')$ and $[a_{ij}(\bar{\boldsymbol{K}}'')]''$ are formally identical to the corresponding ones in a medium with magnetic isotropy ($\boldsymbol{\lambda} = \mathbf{1}$) but with ϵ_i replaced by ϵ''_i.

(ii) We can also pass on from the X^0 frame to the $\boldsymbol{\epsilon}$ frame by a transformation $X^0 = \hat{\mathbf{R}}\hat{X}$. While $\bar{\mathbf{R}}$ is the transformation from the X^0 frame to the $\boldsymbol{\mu}$ frame, $\hat{\mathbf{R}}$ is the corresponding transformation from the X^0 frame to the $\boldsymbol{\epsilon}$ frame. (Henceforward, all quantities referred to the $\boldsymbol{\mu}$ and $\boldsymbol{\epsilon}$ frames will be indicated by the superposed signs ($\bar{}$) and ($\hat{}$) respectively.) With the substitutions $\hat{\boldsymbol{\lambda}}' = \hat{\mathbf{D}}\hat{\boldsymbol{\lambda}}\,\hat{\mathbf{D}}$, where $\hat{D}_{ij} = 0$ for $i \neq j$, $\hat{D}_{ii} = \sqrt{\epsilon_i}$, and $\hat{\mathbf{D}}\hat{\boldsymbol{K}} = |\boldsymbol{\epsilon}|^{1/2}\hat{\boldsymbol{K}}'$, we obtain a symmetric matrix $\hat{\boldsymbol{\lambda}}'$ which can be diagonalized by an orthogonal matrix $\hat{\mathbf{S}}$, by $\boldsymbol{\lambda}'' = \hat{\mathbf{S}}^{-1}\hat{\boldsymbol{\lambda}}'\hat{\mathbf{S}}$. Since $\hat{\boldsymbol{K}}' = \hat{\mathbf{S}}\hat{\boldsymbol{K}}''$, we have

$$\frac{|\boldsymbol{\mu}|}{|\boldsymbol{\epsilon}|}A''(\hat{\boldsymbol{K}}'') = \left(\hat{K}''^2 - \sum \mu''_\gamma\right)\sum \mu''_\alpha \hat{K}''^2_\alpha + \sum \mu''^2_\alpha \hat{K}''^2_\alpha + |\boldsymbol{\mu}''| = f(\boldsymbol{\mu}'', \hat{\boldsymbol{K}}'') \tag{6}$$

where we have employed the symbol μ''_i for $1/\lambda''_i$. Again,

$$[a_{ij}(\hat{\boldsymbol{K}}'')]'' = \left(|\boldsymbol{\lambda}''|\sum \frac{\hat{K}''^2_\alpha}{\lambda''_\alpha} + \lambda''_i + \lambda''_j - \sum \lambda''_\alpha\right)\hat{K}''_i\hat{K}''_j - (\lambda''_i\lambda''_j)^{1/2}\delta_{ij}\hat{K}''^2 + \delta_{ij}. \tag{7}$$

It is easily seen that these expressions have attained their normal forms of electric isotropy ($\boldsymbol{\epsilon} = \mathbf{1}$) but with λ_i replaced by λ''_i.

ϵ''_i, λ''_i, $\bar{\mathbf{S}}$ and $\hat{\mathbf{S}}$ are characteristic of the medium, depending only on the principal values ϵ_i and λ_i of the two tensors and the orthogonal matrix $\mathbf{P}$ that connects their principal frames. $\hat{\boldsymbol{\lambda}}'$ and $\bar{\boldsymbol{\epsilon}}'$ are seen to be connected by the similarity transformation $\hat{\boldsymbol{\lambda}}' = \mathbf{Q}\bar{\boldsymbol{\epsilon}}'\mathbf{Q}^{-1}$ with $\mathbf{Q} = \hat{\mathbf{D}}\mathbf{P}^{-1}\bar{\mathbf{D}}$. Hence it follows that they have the same eigenvalues $\lambda''_i = \epsilon''_i = 1/\mu''_i$. Thus, these three constants ϵ''_i decide the nature of electromagnetic wave propagation in the general biaxial medium.

Since the expressions for $A(\boldsymbol{K})$ and $[a_{ij}(\boldsymbol{K})]$ in the general case are reduced formally to magnetic isotropy on applying the set of transformations (i) and to electric isotropy on applying the set (ii), the two sets of transformations can be considered dual to one another; and any theorem arising out of one set, will have its dual arising out of the other.

The energy radiated per unit path length is given by the integral

$$F = \frac{\mathrm{i}\,e^2\beta^2\gamma^2}{8\pi^3}\int \frac{K_1[a_{11}(\boldsymbol{K})]\,\mathrm{d}^3\boldsymbol{K}}{A(\boldsymbol{K})}. \tag{8}$$

Applying the sets (i) and (ii) of transformations respectively, the radiated energy reduces to

$$F = \frac{\mathrm{i}\, e^2\beta^2\gamma^2}{8\pi^3}\int \frac{\Sigma(\lambda_\alpha\lambda_\beta)^{1/2}\bar{R}_{1\alpha}\bar{S}_{\alpha i}\bar{R}_{1\beta}\bar{S}_{\beta j}[a_{ij}(\bar{\boldsymbol{K}}'')]''K_1\,\mathrm{d}^3\boldsymbol{K}}{f(\boldsymbol{\epsilon}'',\bar{\boldsymbol{K}}'')} \tag{9a}$$

$$= \frac{\mathrm{i}\, e^2\beta^2\gamma^2}{8\pi^3}\int \frac{|\boldsymbol{\mu}|}{|\boldsymbol{\epsilon}|}\frac{\Sigma(\eta_\alpha\eta_\beta)^{1/2}\hat{R}_{1\alpha}\hat{S}_{\alpha i}\hat{R}_{1\beta}\hat{S}_{\beta j}[a_{ij}(\hat{\boldsymbol{K}}'')]''K_1\,\mathrm{d}^3\boldsymbol{K}}{f(\boldsymbol{\mu}'',\hat{\boldsymbol{K}}'')}, \tag{9b}$$

where $\boldsymbol{\eta}$ is the reciprocal permittivity.

The shape of the contour for the K_3 integration is ascertained by determining the sign of the imaginary part ξ of K_3. Let e_i and m_i be the infinitesimal imaginary parts in ϵ_i and μ_i respectively. The equation giving the poles of the integrand in the K_3 plane is therefore obtained as

$$\left(\sum_i e_i\frac{\partial f}{\partial \epsilon_i}+\sum_i m_i\frac{\partial f}{\partial \mu_i}\right)+\xi\frac{\partial f}{\partial K_3}=0. \tag{10}$$

Now,

$$\frac{\partial f}{\partial \epsilon_i}=\frac{\partial f(\boldsymbol{\epsilon}'',\bar{\boldsymbol{K}}'')}{\partial \epsilon_j''}\frac{\partial \epsilon_j''}{\partial \epsilon_i}=[a_{jj}(\bar{\boldsymbol{K}}'')]''\frac{\partial \epsilon_j''}{\partial \epsilon_i}.$$

Again,

$$\boldsymbol{\epsilon}''=\bar{\mathbf{S}}^{-1}\bar{\boldsymbol{\epsilon}}'\bar{\mathbf{S}}=\bar{\mathbf{S}}^{-1}\,\bar{\mathbf{D}}\bar{\boldsymbol{\epsilon}}\,\bar{\mathbf{D}}\bar{\mathbf{S}}=\bar{\mathbf{S}}^{-1}\,\bar{\mathbf{D}}\mathbf{P}\hat{\boldsymbol{\epsilon}}\mathbf{P}^{-1}\,\bar{\mathbf{D}}\bar{\mathbf{S}}$$

where $\hat{\boldsymbol{\epsilon}}$ is the diagonal permittivity tensor. Since $\bar{\mathbf{S}}$ and $\mathbf{P}$ are orthogonal matrices and $\bar{\mathbf{D}}$ is a diagonal matrix, $(\bar{\mathbf{S}}^{-1}\bar{\mathbf{D}}\mathbf{P})^T=\mathbf{P}^{-1}\,\bar{\mathbf{D}}\bar{\mathbf{S}}$. Calling $\bar{\mathbf{S}}^{-1}\,\bar{\mathbf{D}}\mathbf{P}=\bar{\mathbf{T}}$, we thus have $\epsilon_j''=\bar{T}_{ji}^2\epsilon_i$, and

$$\frac{\partial f}{\partial \epsilon_i}=\sum_j \bar{T}_{ji}^2[a_{jj}(\bar{\boldsymbol{K}}'')]''.$$

To evaluate the second term in (10) we pass on to the $\boldsymbol{\mu}$ frame and note that

$$f(\boldsymbol{\epsilon}'',\bar{\boldsymbol{K}}'')\to f(\boldsymbol{\mu}'',\hat{\boldsymbol{K}}'')$$

when $\boldsymbol{\epsilon}''\to\boldsymbol{\mu}''$ and $\bar{\boldsymbol{K}}''\to\hat{\boldsymbol{K}}''$. Equation (10) thus simplifies to

$$\sum_i\left[e_i\left(\sum_j \bar{T}_{ji}^2[a_{jj}(\bar{\boldsymbol{K}}'')]''\right)+\frac{|\boldsymbol{\epsilon}|^2}{|\boldsymbol{\mu}|^2}m_i\left(\sum_j \hat{T}_{ji}^2\{a_{jj}(\hat{\boldsymbol{K}}'')\}''\right)\right]+\xi\sum_{i,j}\frac{\hat{R}_{3j}\hat{S}_{ji}}{\sqrt{\eta_j}}\frac{\partial f(\boldsymbol{\mu}'',\hat{\boldsymbol{K}}'')}{\partial \hat{K}_i''}=0 \tag{11}$$

where $\hat{\mathbf{T}}=\hat{\mathbf{S}}^{-1}\,\hat{\mathbf{D}}^{-1}\mathbf{P}^{-1}$ and $\{a_{jj}(\hat{\boldsymbol{K}}'')\}''$ are obtained from $[a_{jj}(\bar{\boldsymbol{K}}'')]''$ by the substitutions $\epsilon_i''\to\mu_i''$ and $\bar{K}_i''\to\hat{K}_i''$ in the latter.

The parametrization of $f(\boldsymbol{\epsilon}',\boldsymbol{K}')$ and $[a_{jj}(\boldsymbol{K}')]'$ in the simpler case of coincident principal frames of $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ is given by Majumdar (1973). Since $f(\boldsymbol{\epsilon}',\boldsymbol{K}')\to f(\boldsymbol{\epsilon}'',\bar{\boldsymbol{K}}'')$ and $[a_{jj}(\boldsymbol{K}')]'\to[a_{jj}(\bar{\boldsymbol{K}}'')]''$ as $\epsilon_i'\to\epsilon_i''$ and $K_i'\to\bar{K}_i''$, we note that an identical set of parameters can be introduced by replacing ϵ_i' by ϵ_i'' and K_i' by $\bar{K}_i''$ in his expressions. The same substitutions once again serve to achieve the parametrization of $f(\boldsymbol{\mu}'',\hat{\boldsymbol{K}}'')$ and $\{a_{jj}(\hat{\boldsymbol{K}}'')\}''$ if we replace ϵ_i'' by μ_i'' and $\bar{K}_i''$ by $\hat{K}_i''$. However, the assumption that $\epsilon_1''>\epsilon_2''>\epsilon_3''$ implies $\mu_1''<\mu_2''<\mu_3''$, and thus there is an interchange of expressions on the inner and outer sheets when we go from the $\boldsymbol{\epsilon}$ frame to the $\boldsymbol{\mu}$ frame. It then follows that similar arguments apply even in this general case and the shape of the contour for K_3 integration remains unaltered.

We now construct an orthonormal triplet of vectors $\bar{R}''_{1\alpha}, \bar{R}''_{2\alpha}, \bar{R}''_{3\alpha}$ where $\bar{\mathbf{R}}'' = \bar{\mathbf{R}}'\bar{\mathbf{S}}$ and

$$\bar{R}'_{1i} = \left(\frac{\lambda_i}{\lambda_{11}}\right)^{1/2} \bar{R}_{1i}, \qquad \bar{R}'_{2i} = \left(\frac{\lambda_i \lambda_{11}}{|\boldsymbol{\lambda}|\mu_{33}}\right)^{1/2} \left(\bar{R}_{2i} - \frac{\lambda_{12}}{\lambda_{11}} \bar{R}_{1i}\right), \qquad \bar{R}'_{3i} = (\lambda_i \mu_{33})^{-1/2} \bar{R}_{3i}. \tag{12}$$

Noting that K_1 is held fixed while the integration over K_2 is carried out, the integral (9*a*) reduces to

$$F = -\frac{e^2\beta^2\gamma^2}{8\pi^2} \int \left(\frac{\lambda_{11}}{|\boldsymbol{\lambda}|}\right)^{1/2} \left(\frac{\bar{R}''_{1i}\bar{R}''_{1j}[a_{ij}(\bar{K}'')]'' K_1\, \mathrm{d}K_1 (\Sigma \bar{R}''_{2i} \mathrm{d}\bar{K}''_i)}{\Sigma_i \bar{R}''_{3i}\, \partial f(\boldsymbol{\epsilon}'', \bar{K}'')/\partial \bar{K}''_i}\right). \tag{13}$$

If one now makes the substitutions $\boldsymbol{\lambda} = \mathbf{1}$, $\bar{\mathbf{R}}'' \to \hat{\mathbf{R}}$, $\bar{K}''_i \to \hat{K}_i$, and $\epsilon''_i \to \epsilon_i$, the resulting integrand is exactly that obtained in the simpler case of magnetic isotropy. It follows that if in the expression for the radiated energy in a medium with $\boldsymbol{\lambda} = \mathbf{1}$, we make the substitutions $\hat{\mathbf{R}} \to \bar{\mathbf{R}}''$, $\epsilon_i \to \epsilon''_i$, $K_1 \to (|\boldsymbol{\lambda}|/\lambda_{11})^{1/2} K_1$ and multiply the result by $(\lambda_{11}/|\boldsymbol{\lambda}|)^{1/2}$, we obtain the radiated energy for a charge moving in an arbitrary direction in a general doubly anisotropic medium. Since the integral for the field differs from that of the energy loss in the presence of an exponential factor in the numerator, an additional substitution is required in the case of the field, namely, $\hat{x}_i \to \bar{x}''_i$ where

$$x^0_i = \bar{R}_{ij}\bar{x}_j = \sum \left(\frac{|\boldsymbol{\lambda}|}{\lambda_j}\right)^{1/2} \bar{R}_{ij}\bar{S}_{jk}\bar{x}''_k. \tag{14}$$

The dual of this theorem is obtained if one proceeds from (9*b*) and applies the set (ii) of transformations. By symmetry, we can infer that the scheme of substitutions that takes us from electric isotropy to double anisotropy is

$$\bar{\mathbf{R}} \to \hat{\mathbf{R}}'' = \hat{\mathbf{R}}'\hat{\mathbf{S}}, \qquad \lambda_i \to \lambda''_i, \qquad K_1 \to \left(\frac{|\boldsymbol{\eta}|}{\eta_{11}}\right)^{1/2} K_1, \qquad \bar{x}_i \to \hat{x}''_i$$

and an overall multiplication factor of $(\eta_{11}/|\boldsymbol{\eta}|)^{1/2}$. $\hat{\mathbf{R}}'$ is obtained from $\bar{\mathbf{R}}'$ of (12), and $\hat{x}''_i$ from $\bar{x}''_i$ of (14) by the substitutions

$$\bar{\mathbf{R}} \to \hat{\mathbf{R}}, \qquad \lambda_i \to \eta_i, \qquad \mu_i \to \epsilon_i, \qquad \bar{\mathbf{S}} \to \hat{\mathbf{S}}.$$

3. Energy loss in a generalized uniaxial medium

Let us apply the above theorem to calculate the energy loss of a charge moving uniformly in an arbitrary direction in a generalized uniaxial medium with $\epsilon''_2 = \epsilon''_3 (\mu''_2 = \mu''_3)$. We start with the well known expression for the radiated energy in a uniaxial ferrite with electric isotropy ($\boldsymbol{\epsilon} = \mathbf{1}$, $\mu_2 = \mu_3$),

$$F_i = \frac{e^2}{4\pi c^2} \int \left[\left(\frac{\Sigma_i \lambda_i \bar{R}^2_{1i}}{|\boldsymbol{\lambda}|}\right)^{1/2} - K_1^2\right] \omega\, \mathrm{d}\omega. \tag{15}$$

Under scheme (ii) of substitutions,

$$\sum_i \lambda_i \bar{R}_{1i}^2 \to \sum_i \lambda_i'' \hat{R}_{1i}''^2 = \sum_i \frac{(\eta_j\eta_k)^{1/2}}{\eta_{11}} \hat{R}_{1j}\hat{S}_{ji}\hat{R}_{1k}\hat{S}_{ki}\lambda_i''. \tag{16}$$

Now, $\boldsymbol{\lambda}'' = \hat{\mathbf{S}}^{-1}\,\hat{\mathbf{D}}\mathbf{P}^{-1}\bar{\boldsymbol{\lambda}}\mathbf{P}\,\hat{\mathbf{D}}\hat{\mathbf{S}}$ and hence

$$\lambda_i'' = (\epsilon_\rho\epsilon_\nu)^{1/2}\hat{S}_{\rho i}\hat{S}_{\nu i}P_{\mu\rho}P_{\mu\nu}\lambda_\mu. \tag{17}$$

Substituting equation (17) in (16), we have,

$$\sum_i \lambda_i \bar{R}_{1i}^2 \to \sum_i \frac{(\eta_j\eta_k)^{1/2}}{\eta_{11}} \hat{R}_{1j}\hat{S}_{ji}\hat{R}_{1k}\hat{S}_{ki}\hat{S}_{\rho i}\hat{S}_{\nu i}P_{\mu\rho}P_{\mu\nu}(\epsilon_\rho\epsilon_\nu)^{1/2}\lambda_\mu = \frac{\hat{R}_{1j}\hat{R}_{1k}P_{\mu j}P_{\mu k}\lambda_\mu}{\eta_{11}} = \frac{\lambda_{11}}{\eta_{11}}. \tag{18}$$

Thus,

$$F_i \to \frac{e^2}{4\pi c^2}\int\left[\left(\frac{\lambda_{11}|\boldsymbol{\eta}|}{\eta_{11}|\boldsymbol{\lambda}|}\right)^{1/2} - K_1^2\frac{|\boldsymbol{\eta}|}{\eta_{11}}\right]\left(\frac{\eta_{11}}{|\boldsymbol{\eta}|}\right)^{1/2}\omega\,\mathrm{d}\omega. \tag{19}$$

In terms of the ordinary ϵ (and not their multiples $(\omega^2/c^2)\epsilon$), we have

$$F = \frac{e^2}{4\pi c^2}\int\left(\sqrt{[\mu_{11}]} - \frac{1}{\beta^2\sqrt{[\epsilon_{11}]}}\right)\omega\,\mathrm{d}\omega. \tag{20}$$

It is interesting to note that the energy loss can be expressed in this form where it depends only on the tensor components of $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ and does not involve the auxiliary matrices $\bar{\mathbf{S}}$ and $\hat{\mathbf{S}}$, and the constants ϵ_i''. The same result is also obtained if we start with the expression for the radiated energy in a uniaxial dielectric with magnetic isotropy ($\epsilon_2 = \epsilon_3, \boldsymbol{\lambda} = \mathbf{1}$) and apply the scheme (i) of substitutions.

In terms of K_i, the equation $f(\boldsymbol{\epsilon}'', \bar{\boldsymbol{K}}'') = f(\boldsymbol{\mu}'', \hat{\boldsymbol{K}}'') = 0$ represents the Fresnel wave surface. When $\epsilon_2'' = \epsilon_3''$, $f(\boldsymbol{\epsilon}'', \bar{\boldsymbol{K}}'')$, for instance, factorizes as follows:

$$f(\boldsymbol{\epsilon}'', \bar{\boldsymbol{K}}'') = (\bar{K}''^2 - \epsilon_2'')\left(\sum \epsilon_\alpha''\bar{K}_\alpha''^2 - \epsilon_1''\epsilon_2''\right). \tag{21}$$

Transforming to the unaccented variables, and noting that

$$\sum_\alpha \epsilon_\alpha''\bar{K}_\alpha''^2 = \bar{S}_{i\alpha}\bar{\epsilon}_{ij}'\bar{S}_{j\alpha}\,\bar{S}_{l\alpha}\bar{K}_l'\,\bar{S}_{m\alpha}\bar{K}_m' = \bar{\epsilon}_{ij}'\bar{K}_i'\bar{K}_j' = |\boldsymbol{\lambda}|\epsilon_{\alpha\beta}K_\alpha K_\beta$$

we obtain

$$([\lambda_{\alpha\beta}]K_\alpha K_\beta - \epsilon_2'')(|\boldsymbol{\lambda}|\epsilon_{\alpha\beta}K_\alpha K_\beta - \epsilon_1''\epsilon_2'') = 0. \tag{22}$$

The equation of the Čerenkov cone is obtained from this by the transformations $K_1 \to \gamma K_1, \epsilon_{ij} \to \beta^2\gamma^2K_1^2\epsilon_{ij}, K_i \to x_i^0$ and omitting γ. This leads to two distinct cones whose equations are

$$\epsilon_{\alpha\beta}x_\alpha^0x_\beta^0 - \beta^2\frac{|\boldsymbol{\epsilon}|}{\epsilon_2''}x_1^{02} = 0 \tag{23a}$$

and

$$\mu_{\alpha\beta}x_\alpha^0x_\beta^0 - \beta^2\frac{|\boldsymbol{\mu}|}{\mu_2''}x_1^{02} = 0. \tag{23b}$$

The characteristic cones which are reciprocal to the radiation cones have the equations

$$[\epsilon_{\alpha\beta}]x_\alpha^0 x_\beta^0 - \beta^2 \frac{|\epsilon|}{\epsilon_2''}(\epsilon_{33}x_2^{02} + \epsilon_{22}x_3^{02} - 2\epsilon_{23}x_2^0 x_3^0) = 0 \tag{24a}$$

and

$$[\mu_{\alpha\beta}]x_\alpha^0 x_\beta^0 - \beta^2 \frac{|\mathbf{\mu}|}{\mu_2''}(\mu_{33}x_2^{02} + \mu_{22}x_3^{02} - 2\mu_{23}x_2^0 x_3^0) = 0. \tag{24b}$$

Exactly the same equations can be obtained from the equation $f(\mathbf{\mu}'', \hat{\mathbf{K}}'') = 0$ and applying the other set of transformations.

The results of this section reduce to the corresponding ones in various less general situations. When the medium is uniaxial by virtue of parallel principal frames and $\mu_2 = \mu_3$, $\epsilon_2 = \epsilon_3$ we obtain the expressions of Muzikar and Pafamov (1961). For parallel principal frames, but with the less restrictive condition $\epsilon_2\mu_3 = \epsilon_3\mu_2$, the results of Majumdar and Pal (1970) follow.

Acknowledgment

The author is deeply indebted to Dr S Datta Majumdar for the central idea of this work and indispensable guidance at several crucial points.

References

Born M and Wolf E 1965 *Principles of Optics* (Oxford: Pergamon) p 665
Gerritsen H J, Harrison S E and Lewis H R 1960 *J. appl. Phys.* **31** 1566
Jelley J V 1958 *Čerenkov Radiation and its Applications* (Oxford: Pergamon) chap 3
Landau L D and Lifshitz E M 1960 *Electrodynamics of Continuous Media* (Oxford: Pergamon) p 252
Lashinsky H 1956 *J. appl. Phys.* **27** 631
Lewandowski S J 1971 *J. Phys. A: Gen. Phys.* **4** 197
Majumdar S D 1973 *Ann. Phys., NY* **76** 428
Majumdar S D and Pal R 1970 *Proc. R. Soc.* A **316** 525
—— 1973 *Ann. Phys., NY* **76** 419
Muzikar C and Pafamov V E 1961 *Czech. J. Phys.* **11** 709
O'Dell T H 1970 *Selected Topics in Solid State Physics* vol 11, ed E P Wohlfarth (Amsterdam and London: North-Holland) p 83